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ONE-DIMENSIONAL FINALLY PRECONTINUOUS PSEUDOREPRESENTATIONS OF CONNECTED LOCALLY COMPACT GROUPS

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Abstract. The structure of one-dimensional finally precontinuous pseudo-representations of connected locally compact groups is studied.

For generalities concerning quasirepresentations, pseudorepresentations, pseudocharacters, and quasicharacters, see [1–3].

§ 1. Introduction

Let us recall the general structure of finite-dimensional quasirepresentations of groups.

Theorem 1 [1]**.** *Let G be a group and let π be a quasirepresentation of the group G in a finite-dimensional vector space E_π. Let E_π^* be the space dual to E_π. Let L be the set of vectors $\xi \in E_T$ such that the orbit $\{\pi(g)\xi \mid g \in G\}$ is bounded in E; let M be the set of functionals $f \in E_\pi^*$ such that the orbit $\{\pi(g)^* f \mid g \in G\}$ is bounded in E_π^*. In this case, both the set L and the annihilator $M^\perp$ of M are π-invariant vector subspaces of E_π. Consider the ascending family of subspaces $\{0\}$, $L \cap M^\perp$, $M^\perp$, $L + M^\perp$, and $E = E_\pi$ and write out the matrix $p(g)$ of the operator $\pi(g)$, $g \in G$, in the block form corresponding to the decomposition of the space E into the direct sum of the*

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subspaces $L\cap M^{\perp}$, $M^{\perp}\setminus(L\cap M^{\perp})$, $L\setminus(L\cap M^{\perp})$, and $E\setminus(L+M^{\perp})$, where the symbol "$\setminus$" means the passage to a complementary subspace,

$$(*)\qquad p(g)=\begin{pmatrix}\alpha(g) & \varphi(g) & \sigma(g) & \tau(g)\\ 0 & \beta(g) & 0 & \rho(g)\\ 0 & 0 & \gamma(g) & \chi(g)\\ 0 & 0 & 0 & \delta(g)\end{pmatrix},\qquad g\in G.$$

(Here we have $p_{23}(g)=0$ because L is π-invariant.) Then the following assertions hold:

(1) *the mappings α, δ, γ, σ, and χ are bounded;*

(2) *the matrix-valued mappings p_1 and p_2 defined by the relations*

$$p_1(g)=\begin{pmatrix}\alpha(g) & \varphi(g)\\ 0 & \beta(g)\end{pmatrix}$$

and

$$p_2(g)=\begin{pmatrix}\beta(g) & \rho(g)\\ 0 & \delta(g)\end{pmatrix}$$

are representations of G.

In this paper, we consider the structure of one-dimensional pseudorepresentations with sufficiently small defect of connected locally compact groups. If π is a pseudorepresentation of this kind, then π must coincide with one of the diagonal blocks of the above matrix. Therefore, π is either an ordinary one-dimensional representation of G (bounded or unbounded) or a bounded one-dimensional pseudorepresentation of the group. Below we describe the structure of bounded one-dimensional finally precontinuous pseudorepresentations with small defect of connected locally compact groups.

§ 2. Preliminaries

We also recall the following notion.

Lemma 1. *Let G be an almost connected locally compact group, and let $\mathcal{N}$ be the family of compact normal subgroups $N\neq\{e\}$ such that G/N is a (not necessarily connected) Lie group. Then $\mathcal{N}$ is a nontrivial filter basis convergent to $\{e\}$.*

Proof. This follows immediately from the Gleason–Montgomery–Zippin–Yamabe theorem.

Recall that a one-dimensional locally bounded (in particular, bounded) pseudorepresentation of G is said to be *finally precontinuous* if the related set $\mathrm{FDG}(\pi) = \bigcap_{N\in\mathcal{N}} \overline{\pi(N)}$ is the singleton formed by the number 1 [1]. As is known, this condition implies that the restriction of π to the commutator subgroup G' of G is continuous with respect to the intrinsic Lie topology of the Lie group G' [1, 4].

Recall that a pseudorepresentation is said to be pure if its restriction to every commutative subgroup is an ordinary representation of the subgroup.

§ 3. Main theorem

Let us describe the structure of bounded one-dimensional finally precontinuous pseudorepresentations with sufficiently small defect of connected locally compact groups.

Theorem 2. *Let G be a connected locally compact group and let π be a one-dimensional finally precontinuous pseudorepresentation of G with sufficiently small defect* [5]*. Let N be a compact normal subgroup of G such that the quotient G/N is a Lie group and the restriction of π to N is the identity representation of N. Let $G/N = SR$ be a Levi decomposition, where R is the radical of G/N and S stands for a Levi subgroup of G/N. Denote the representation of G/N defined by the representation π of G by π again. Then π is the one-dimensional pseudorepresentation corresponding to the one-dimensional quasirepresentation given by the product of a* (*not necessarily continuous*) *central unitary character φ of R* (*i.e., $\varphi(k) = \varphi(gkg^{-1})$ for all $k \in R$ and $g \in G/N$*) *and a Guichardet–Wigner one-dimensional representation of S* (*i.e., a mapping of the form*

$$s \mapsto \exp(ir\chi(s)), \qquad s \in S;$$

here $r \in \mathbb{R}$ and χ is a Guichardet–Wigner pseudocharacter on S, see [1, 3]).

Proof. Since every solvable group is amenable, and every one-dimensional pseudorepresentation of an amenable group is obviously an ordinary representation of the group [1], it follows that the restriction φ of the one-dimensional pseudorepresentation π of G/N to R is an ordinary representation of R, i.e., a character of R. Repeating the argumentation used in the proof of the generalization of Lie's theorem concerning finite-dimensional representations of solvable locally divisible groups in [6], we can see that this character is central.

The restriction χ of π to S is automatically continuous by Theorem 3.3.1 of [1]. This restriction is an ordinary character on every solvable subgroup of S, including the solvable subgroup AN of a generalized Iwasawa decomposition $S = KAN$ of S and the corresponding subgroup of every $\mathfrak{sl}(2,\mathbb{R})$-triple system. Since the restriction of π to the latter subgroup is close to its inverse, it follows that all these restrictions are the identity characters of the corresponding subgroups, and therefore the restriction of χ to AN is also the identity character of AN. Therefore, χ is completely determined by its restriction to K. Since simple compact subgroups of K have only trivial continuous characters, it follows that the (continuous) restriction of χ to K is an ordinary continuous character of the part of K whose Lie algebra $\mathfrak{l}$ is Abelian. However, this character is the exponential of an ordinary linear functional on $\mathfrak{l}$, which precisely corresponds to the pseudocharacter defined step-by-step as the Guichardet–Wigner pseudocharacter on S, see Definition 2.5.12 and Lemma 3.3.10 of [1]).

Finally, let $\Phi(g) = \chi(s)\varphi(r)$ for $g \in sr$, $s \in S$, $r \in R$, $g = sr$; then $\Phi(s_1r_1)\Phi(s_2r_2) = \chi(s_1)\varphi(r_2)\chi(s_2)\varphi(r_2) = \chi(s_1)\chi(s_2)\chi(s_2^{-1})\varphi(r_2)\chi(s_2)\varphi(r_2)$ for $s_1, s_2 \in S$ and $r_1, r_2 \in R$, and this is close to $\chi(s_1s_2)\varphi(s_2^{-1}r_2s_2r_2) = \Phi(s_1r_1s_2r_2)$. This shows that Φ is a one-dimensional quasirepresentation close to the one-dimensional pseudorepresentation π, and hence π is obtained from Φ by the standard approximation procedure on every cyclic subgroup of G (see [5, 7, 8]).

§ 4. Concluding remarks

By [5], we have

Theorem 3. *If the defect of a one-dimensional pseudorepresentation π is less than $\sqrt{3}$ (i.e., $|\pi(gh) - \pi(g)\pi(h)| \le q < \sqrt{3}$ for some q and all $g, h \in G$), then π is pure.*

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